Exact Solutions in Locally Anisotropic Gravity and Strings

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Abstract

In this Report we outline some basic results on generalized Finsler–Kaluza–Klein gravity and locally anisotropic strings. There are investigated exact solutions for locally anisotropic Friedmann–Robertson–Walker universes and three dimensional and string black holes with generic anisotropy.

Devoted to the memory of Professor Ryszard Raczka (1931–1996)

1   Introduction

The theory of locally anisotropic field interactions and (super)strings is recently descussed [6, 7, 8, 9] in the context of development of unified approaches to generalized Finsler like [3] and Kaluza–Klein gravity [4]. A number of present day cosmological models are constructed as higher–dimensional extensions of general relativity with a general anisotropic distribution of matter and in correlation with low–energy limits of string perturbation theory. In non–explicit form it is assumed the **postulate: the matter always (even being anisotropic) gives rise to a locally isotropic geometry,** which is contained in the structure of Einstein equations for metric $g_{ij}(x^k)$

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on (pseudo)Riemannian spaces:

$$G_{ij}(x^k) \qquad \simeq \qquad T_{ij}(x^k, y^a)$$

Einstein tensor $\qquad\qquad$ Energy–momentum tensor

$\boxed{\text{(for a locally isotropic curved space)}}$ $\qquad$ $\boxed{\text{(in general anisotropic)}}$

where $x^i, i = 0, 1, ..., n - 1$ are coordinates on space–time M and $y^a, a = 1, 2, ..., m$ are parameters (coordinates) of anisotropies.

Anisotropic cosmological and locally anisotropic self–gravitating models are widely used in order to interpret the observable anisotropic structure of the Universe and of background radiation. Our basic idea to be developed in this paper is that cosmological anisotropies are not only consequences of some anisotropic distributions of matter but they reflect a generic space–time anisotropy induced after reductions from higher to lower dimensions and by primordial quantum field fluctuations. If usual Kaluza–Klein theories routinely require compactification mechanisms, we suggest a more general scenarios of possible decompositions of higher dimensional (super)space into lower dimensional ones being modelled by a specific "splitting field" defined geometrically as a nonlinear connection.

A geometry of manifolds provided with a metric more general than the usual Riemann one, $g_{ij}(x^k) \implies g_{ij}(x^k, \lambda^s y^n)$, where $y^n \simeq \frac{dx^n}{dt}$ and λ^s is a parameter of homogeneity of order s, was proposed in 1854 by B. Riemann and it was studied for the first time in P. Finsler (1918) and E. Cartan (1934) (see historical overviews, basic results and references in [3, 8, 9]). At first sight there are very substantial objections of physical character to generalized Finsler like theories: One was considered that a local anisotropy crucially frustrates the local Lorentz invariance. Not having even local (pseudo)rotations and translations it is an unsurmountable problem to define conservation laws and values of energy–momentum type, to apply the concept of fundamental particles fields (for example, without local rotations we can not define local groups and algebras and their representations). A difficulty with Finsler like gravity was also the problem of its inclusion into the framework of modern approaches based on (super)strings, Kaluza–Klein and gauge theories.

The main purpose of a series of our works (see [6, 7, 8, 9] and references) is the development of a general approach to locally anisotropic gravity imbedding both type of Kaluza–Klein and Finsler–like theories. It should be emphasized that a subclass of such models can be constructed as to have a local space–time Lorentz invariance. We proved that the general higher order anisotropic gravity can be treated as alternative low energy limits of (super)string theories with a dynamical reduction given by the nonlinear connection field and that there are natural extensions of the Einstein gravity to locally anisotropic theories constructed on generic nonholonomic vector bundles provided with nonlinear connection structure.

The field equations of locally anisotropic gravity are of type

$$\boxed{G_{\alpha\beta}(x^\alpha, y^\beta) \simeq T_{\alpha\beta}(x^\alpha, y^\beta)}$$

where the Einstein tensor is defined on a bundle (generalized Finsler) space, x^α are usual coordinate on the base manifold and y^β are coordinates on the fibers (parameters of anisotropy), in general $\dim\{x^\alpha\} \neq \dim\{y^\beta\}$.

This paper is organized as follows. In Sec. II we briefly review the geometric background of locally anisotropic gravity. Models of locally anisotropic Friedmann–Robertson–Walker universe are considered in Sec. III. In Sec. IV we analyze anisotropic black hole solutions in three dimensional space–times and extend such solutions to the string theory. Conclusions are drawn in Sec. V.

2 Generalized Finsler–Kaluza–Klein gravity

In Einstein gravity and its locally isotropic modifications of Kaluza–Klein, lower dimensional, or of Einstein–Cartan–Weyl types, the fundamental space–time is considered as a real $(4 + d)$–dimensional, where $(d = -2, -1, 0, 1, ..., n)$, manifold of necessary smoothly class and signature, provided with independent metric (equivalently, tetrad) and linear connection (in general nonsymmetric). In order to model spaces with generic local anisotropy instead of manifolds one considers vector, or tangent/cotangent, bundles (with possible higher order generalizations) enabled with nonlinear connection and distinguished (by the nonlinear connection) linear connection and metric structures. The coordinates in fibers are treated as parameters of possible anisotropy and/or as higher dimension coordinates which in general are not compactified.

In this section we outline the basic results from the so–called locally anisotropic (la) gravity [3, 6, 7, 8, 9] (in brief we shall use la–gravity, la–space and so on).

Let $\mathcal{E} = (E, \pi, F, Gr, M)$ be a locally trivial vector bundle (v–bundle) over a base M of dimension n, where $F = \mathcal{R}^m$ is the typical real vector space of dimension m, the structural group is taken to be the group of linear transforms of $\mathcal{R}^m$, i. e. $Gr = GL(m, \mathcal{R})$. We locally parametrize $\mathcal{E}$ by coordinates $u^\alpha = (x^i, y^a)$, where $i, j, k, l, m, ..., = 0, 1, ..., n-1$ and $a, b, c, d, ... = 1, 2, ..., m$. Coordinate transforms $(x^k, y^a) \rightarrow (x^{k'}, y^{a'})$ on $\mathcal{E}$, considered as a differentiable manifold, are given by formulas $x^{k'} = x^{k'}(x^k), y^{a'} = M_a^{a'}(x)y^a$, where $rank(\frac{\partial x^{k'}}{\partial x^k}) = n$ and $M_a^{a'}(x) \in Gr$.

One of the fundamental objects in the geometry of la–spaces is the **nonlinear connection,** in brief **N–connection.** The N-connection can be defined as a global decomposition of v-bundle $\mathcal{E}$ into horizontal, $\mathcal{HE}$, and vertical, $\mathcal{VE}$, subbundles of the tangent bundle $\mathcal{TE}, \mathcal{TE} = \mathcal{HE} \oplus \mathcal{VE}$. With

respect to a N–connection in $\mathcal{E}$ one defines a covariant derivation opera-tor $\nabla_Y A = Y^i \left\{ \frac{\partial A^a}{\partial x^i} + N_i^a(x, A) \right\} s_a$, where s_a are local linearly indepen-dent sections of $\mathcal{E}, \Lambda = \Lambda^a s_a$ and $Y = Y^i s_i$ is the decomposition of a vector field Y with respect to a local basis s_i on M. Differentiable func-tions $N_i^a(x, y)$ are called the coefficients of the N–connection. One holds these transformation laws for components N_i^a under coordinate transforms: $N_{i'}^{a'} \frac{\partial x^{i'}}{\partial x^i} = M_a^{a'} N_i^a + \frac{\partial M_a^{a'}}{\partial x^i} y^a$. The N–connection is also characterized by its **curvature**

$$\Omega_{ij}^a = \frac{\partial N_j^a}{\partial x^i} - \frac{\partial N_i^a}{\partial x^j} + N_j^b \frac{\partial N_i^a}{\partial y^b} - N_i^b \frac{\partial N_j^a}{\partial y^b},$$

and by its linearization which is defined as $\Gamma_{.bi}^a(x, y) = \frac{\partial N_i^a(x,y)}{\partial y^b}$. The usual linear connections $\omega_{.b}^a = K_{.bi}^a(x) dx^i$ in a v–bundle $\mathcal{E}$ form a particular class of N–connections with coefficients parametrized as $N_i^a(x, y) = K_{.bi}^a(x) y^b$.

Having introduced in a v–bundle $\mathcal{E}$ a N–connection structure we must modify the operation of partial derivation and introduce a locally adapted (to the N–connection) basis (frame)

$$\frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a(x, y) \frac{\partial}{\partial y^a}, \frac{\delta}{\delta y^a} = \frac{\partial}{\partial y^a} \right), \qquad (2.1)$$

instead of the local coordinate basis $\frac{\partial}{\partial u^a} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} \right)$. The basis dual to $\frac{\delta}{\delta u^\alpha}$ is written as

$$\delta u^\alpha = (\delta x^i = dx^i, \delta y^a = dy^a + N_i^a(x, y) dx^i). \qquad (2.2)$$

We note that a v–bundle provided with a N–connection structure is a generic nonholonomic manifiold because in general the nonholonomy coeffi-cients $w_{.\alpha\beta}^\gamma$, defined by relations $[\frac{\delta}{\delta u^\alpha}, \frac{\delta}{\delta u^\beta}] = \frac{\delta}{\delta u^\alpha} \frac{\delta}{\delta u^\beta} - \frac{\delta}{\delta u^\beta} \frac{\delta}{\delta u^\alpha} = w_{.\alpha\beta}^\gamma \frac{\delta}{\delta u^\gamma}$, do not vanish.

By using bases (2.1) and (2.2) we can introduce the algebra of tensor distinguished fields (d–fields, d–tensors) on $\mathcal{E}, \mathcal{C} = \mathcal{C}_{qs}^{pr}$, which is equivalent to the tensor algebra of the v–bundle $\mathcal{E}_d$ defined as $\pi_d : \mathcal{H}\mathcal{E} \oplus \mathcal{V}\mathcal{E} \to \mathcal{T}\mathcal{E}$ [3]. An element $t \in \mathcal{C}_{qs}^{pr}$, of d–tensor of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, are written in local form as

$$t = t_{j_1 \ldots j_q b_1 \ldots b_s}^{i_1 \ldots i_p a_1 \ldots a_r}(u) \frac{\delta}{\delta x^{i_1}} \otimes \ldots \otimes \frac{\delta}{\delta x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_p} \otimes$$

$$\frac{\partial}{\partial y^{a_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{a_r}} \otimes \delta y^{b_1} \otimes \ldots \otimes \delta y^{b_s}.$$

In addition to d–tensors we can consider different types of d-objects with group and coordinate transforms adapted to a global splitting of v-bundle by a N–connection.

A **distinguished linear connection,** in brief **a d–connection,** is de-fined as a linear connection D in $\mathcal{E}$ conserving as a parallelism the Whitney

sum $\mathcal{HE} \oplus \mathcal{VE}$ associated to a fixed N-connection structure in $\mathcal{E}$. Components $\Gamma^\alpha_{.\beta\gamma}$ of a d–connection D are introduced by relations $D_\gamma(\frac{\delta}{\delta u^\beta}) = D_{(\frac{\delta}{\delta u^\gamma})}(\frac{\delta}{\delta u^\beta}) = \Gamma^\alpha_{.\beta\gamma}(\frac{\delta}{\delta u^\alpha})$.

We can compute in a standard manner but with respect to a locally adapted frame (2.1), the components of **torsion and curvature of a d–connection** D :

$$T^\alpha_{.\beta\gamma} = \Gamma^\alpha_{.\beta\gamma} - \Gamma^\alpha_{.\gamma\beta} + w^\alpha_{.\beta\gamma} \qquad (2.3)$$

and

$$R^{.\alpha}_{\beta.\gamma\delta} = \frac{\delta\Gamma^\alpha_{.\beta\gamma}}{\delta u^\delta} - \frac{\delta\Gamma^\alpha_{.\beta\delta}}{\delta u^\gamma} + \Gamma^\varphi_{.\beta\gamma}\Gamma^\alpha_{.\varphi\delta} - \Gamma^\varphi_{.\beta\delta}\Gamma^\alpha_{.\varphi\gamma} + \Gamma^\alpha_{.\beta\varphi}w^\varphi_{.\gamma\delta}. \qquad (2.4)$$

The global decomposition by a N–connection induces a corresponding invariant splitting into horizontal $D^h_X = D_{hX}$ (h–derivation) and vertical $D^v_X = D_{vX}$ (v–derivation) parts of the operator of covariant derivation $D, D_X = D^h_X + D^v_X$, where $hX = X^i\frac{\delta}{\delta u^i}$ and $vX = X^a\frac{\partial}{\partial y^a}$ are, respectively, the horizontal and vertical components of the vector field $X = hX + vX$ on $\mathcal{E}$.

Local coefficients $\left(L^i_{.jk}(x,y), L^a_{.bk}(x,y)\right)$ of covariant h-derivation D^h are introduced as $D^h_{(\frac{\delta}{\delta x^k})}\left(\frac{\delta}{\delta x^j}\right) = L^i_{.jk}(x,y)\frac{\delta}{\delta x^i}$, $D^h_{(\frac{\delta}{\delta x^k})}\left(\frac{\partial}{\partial y^b}\right) = L^a_{.bk}(x,y)\frac{\partial}{\partial y^a}$ and $D^h_{(\frac{\delta}{\delta x^k})}f = \frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k} - N^a_k(x,y)\frac{\partial f}{\partial y^a}$, where $f(x,y)$ is a scalar function on $\mathcal{E}$.

Local coefficients $\left(C^i_{.jk}(x,y), C^a_{.bk}(x,y)\right)$ of v-derivation D^v are introduced as $D^v_{(\frac{\partial}{\partial y^c})}\left(\frac{\delta}{\delta x^j}\right) = C^i_{.jk}(x,y)\frac{\delta}{\delta x^i}$, $D^v_{(\frac{\partial}{\partial y^c})}\left(\frac{\partial}{\partial y^b}\right) = C^a_{.bc}(x,y)$ and $D^v_{(\frac{\partial}{\partial y^c})}f = \frac{\partial f}{\partial y^c}$.

By straightforward calculations we can express respectively the coefficients of torsion (2.3) and curvature (2.4) [3] via h- and v-components parametrized as $T^\alpha_{\ \beta\gamma} = \{T^i_{.jk}, T^i_{ja}, T^i_{aj}, T^i_{.ja}, T^a_{.bc}\}$ and $R^{.\alpha}_{\beta.\gamma\delta} = \{R^i_{h.jk}, R^a_{b.jk}, P^i_{j.ka}, P^c_{b.ka}, S^i_{j.bc}, S^a_{b.cd}\}$.

The components of the Ricci d–tensor $R_{\alpha\beta} = R^{.\tau}_{\alpha.\beta\tau}$ with respect to locally adapted frame (2.2) are as follows:$R_{ij} = R^k_{i.jk}, R_{ia} = -{}^2P_{ia} = -P^k_{i.ka}, R_{ai} = {}^1P_{ai} = P^b_{a.ib}, R_{ab} = S^c_{a.bc}$. We point out that because, in general, ${}^1P_{ai} \neq^2 P_{ia}$ the Ricci d–tensor is nonsymmetric.

Now, we shall analyze the compatibility conditions of N- and d–connections and metric structures on the v–bundle $\mathcal{E}$. A metric field on $\mathcal{E}$, $G(u) = G_{\alpha\beta}(u)du^\alpha du^\beta$, is associated to a map $G(X,Y) : \mathcal{T}_u\mathcal{E} \times \mathcal{T}_u\mathcal{E} \to R$, parametrized by a non degenerate symmetric $(n+m) \times (n+m)$–matrix with components $\widehat{G}_{ij} = G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \widehat{G}_{ia} = G\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)$ and $\widehat{G}_{ab} = G\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right)$. One chooses a concordance between N–connection and G–metric structures by imposing conditions $G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}\right) = 0$, equivalently, $N^a_i(x,u) = \widehat{G}_{ib}(x,y) \widehat{G}^{ba}(x,y)$, where $\widehat{G}^{ba}(x,y)$ are found to be components of the matrix $\widehat{G}^{\alpha\beta}$ which is the inverse to $\widehat{G}_{\alpha\beta}$. In this case the metric G on $\mathcal{E}$ is defined by two

independent d–tensors, $g_{ij}(x, y)$ and $h_{ab}(x, y)$, and written as

$$G(u) = G_{\alpha\beta}(u)\,\delta u^{\alpha}\delta u^{\beta} = g_{ij}(x, y)\,dx^i \otimes dx^j + h_{ab}(x, y)\,\delta y^a \otimes \delta y^b. \quad (2.5)$$

The d–connection $\Gamma^{\alpha}_{.\beta\gamma}$ is compatible with the d–metric structure $G(u)$ on $\mathcal{E}$ if one holds equalities $D_{\alpha}G_{\beta\gamma} = 0$.

Having defined the d–metric (2.5) in $\mathcal{E}$ we can introduce the scalar curvature of d–connection $\overleftarrow{R} = G^{\alpha\beta}R_{\alpha\beta} = R + S$, where $R = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$.

Now we can write the Einstein equations for la–gravity

$$R_{\alpha\beta} - \frac{1}{2}G_{\alpha\beta}\overleftarrow{R} + \lambda G_{\alpha\beta} = \kappa_1 \mathcal{T}_{\alpha\beta}, \quad (2.6)$$

where $\mathcal{T}_{\alpha\beta}$ is the energy–momentum d–tensor on la–space, κ_1 is the interaction constant and λ is the cosmological constant. We emphasize that in general the d–torsion does not vanish even for symmetric d–connections (because of nonholonomy coefficients $w^{\alpha}_{\beta\gamma}$). So the d–torsion interactions plays a fundamental role on la–spaces. A gauge like version of la–gravity with dynamical torsion was proposed in [10]. We can also restrict our considerations only with algebraic equations for d–torsion in the framework of an Einstein–Cartan type model of la–gravity.

Finally, we note that all presented in this section geometric constructions contain as particular cases those elaborated for generalized Lagrange and Finsler spaces [3], for which a tangent bundle TM is considered instead of a v-bundle $\mathcal{E}$. We also note that the Lagrange (Finsler) geometry is characterized by a metric of type (2.5) with components parametrized as $g_{ij} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}$ $\left(g_{ij} = \frac{1}{2}\frac{\partial^2 \Lambda^2}{\partial y^i \partial y^j}\right)$ and $h_{ij} = g_{ij}$, where $\mathcal{L} = \mathcal{L}(x, y)$ is a Lagrangian $((\Lambda = \Lambda(x, y))$ is a Finsler metric) on TM, see details in [3, 6, 7, 8, 9, 10]. The usual Kaluza–Klein geometry could be obtained for corresponding parametrizations of N–connection and metric structures on the background v–bundle.

3 Anisotropic Friedmann-Robertson-Walker Universes

In this section we shall construct solutions of Einstein equations (2.6) generalizing the class of Friedmann–Robertson–Walker (in brief FRW) metrics to the case of $(n = 4, m = 1)$ dimensional locally anisotropic space. In order to simplify our considerations we shall consider a prescribed N-connection structure of type $N_0^1 = n(t, \theta), N_1^1 = 0, N_2^1 = 0, N_3^1 = 0$, where the local coordinates on the base M are taken as spherical coordinates for the Robertson-Walker model, $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi$, and the anisotropic coordinate is denoted $y^1 \equiv y$.

The la–metric (2.5) is parametrized by the anzats

$$\delta s^2 = ds_{RW}^2 + h_{11}(t, r, \theta, \varphi, y)\delta y^2 \tag{3.1}$$

where the Robertson-Walker like metric ds_{RW}^2 is written as

$$ds_{RW}^2 = -dt^2 + a^2(t, \theta)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta \cdot d\varphi^2\right)\right],$$

$k = -1, 0$ and 1, respectively, for open, flat and closed universes, $H(t, \theta) = \partial a(t, \theta)/\partial t$ is the anisotropic on angle θ (for our model) Hubble parameter, the containing the N-connection coefficients value δy, see (2.2), is of type $\delta y = dy + n(t, \theta)dt$ and coefficients $n(t, \theta)$ and $h_{11}(t, r, \theta, \varphi, y)$ are considered as arbitrary functions, which are prescribed on la–spaces defined as nonholonomic manifolds (in self–consistent dynamical field models one must find solutions of a closed system of equations for N- and d–connection and d–metric structure).

Considering an anisotropic fluctuation of matter distribution of type $\mathcal{T}_{\alpha\beta} = T_{\alpha\beta}^{(a)} + T_{\alpha\beta}^{(i)}$, with nonvanishing anisotropic components $T_{10}^{(a)}(t, r, \theta) \neq 0$ and $T_{20}^{(a)}(t, \theta) \neq 0$ and diagonal isotropic energy-momentum tensor $T_{\alpha\beta}^{(i)} = diag(-\rho, p, p, p, p_{(y)})$, where ρ is the matter density, p and $p_{(y)}$ are respectively pressures in 3 dimensional space and extended space, we obtain from the Einstein equations (2.6) this generalized system of Friedmann equations:

$$\left(\frac{1}{a}\frac{\partial a}{\partial t}\right)^2 = \frac{8\pi G_{(gr)}}{3}\rho - \frac{k}{a^2}, \tag{3.2}$$

$$\frac{1}{a}\frac{\partial^2 a}{\partial t^2} - n(t, \theta)\frac{1}{a}\frac{\partial a}{\partial t} = -\frac{4\pi G_{(gr)}}{3}(\rho + 3p) \tag{3.3}$$

with anisotropic additional relations between nonsymmetric, for la–spaces, Ricci and energy–momentum d–tensors:

$$R_{10} = -n(t, \theta)\left(\frac{kr}{1 - kr^2} + \frac{2}{r}\right) \simeq T_{10}^{(a)}(t, r, \theta),$$

$$R_{20} = -n(t, \theta) \cdot ctg\theta \simeq T_{20}^{(a)}(t, \theta)$$

when $R_{01} = 0$ and $R_{02} = 0$. The $G_{(gr)}$ from (3.2) and (3.3) is the usual gravitational constant from the Einstein theory.

For the locally isotropic FRW model, when $\rho = -p$, the equations (3.2) and (3.3) have an exponential solution of type $a_{FRW}^{(exp)} = a_0 \cdot e^{\omega_\rho \cdot t}$, where $a_0 = const$ and $\omega_\rho = \sqrt{\frac{8\pi G_{(gr)}}{3}\rho}$. This fact is widely applied in modern cosmology.

Substituting (3.2) into (3.3) we obtain the equation

$$\frac{\partial^2 a}{\partial t^2} - n(t, \theta)\frac{\partial a}{\partial t} - \omega_\rho^2 a = 0 \tag{3.4}$$

where the function $a(t, \theta)$ depends on coordinates t and (as on a parameter) θ. Introducing a new variable $u = a \cdot \exp\left[-\frac{1}{2} \int n(t, \theta) dt\right]$ we can rewrite the (3.4) as a parametric equation

$$\frac{d^2 u(t, \theta)}{dt^2} - \tilde{\omega}(t, \theta) u(t, \theta) = 0$$

for $\tilde{\omega}(t, \theta) = \omega_\rho^2 + \left(\frac{n}{2}\right)^2 + \frac{1}{2}\frac{\partial n}{\partial t}$ which admits expressions of the general solution as series (see [2]).

It is easy to construct exact solutions and understand the physical properties of the equations of type (3.4) if the nonlinear connection structure does not depend on time variable, i.e. $n = n(\theta)$. By introducing the new variable $\tau = \omega_\rho t$ and function $a = v \cdot \exp\left(-D_0\left(\theta\right)\tau\right)$, where $D_0\left(\theta\right) = -n(\theta)/2\omega_\rho$, we transform (3.4) into the equation

$$\frac{d^2 v}{d\tau^2} + \left(1 - D_0^2\left(\theta\right)\right) v = 0$$

which can be solved in explicit form:

$$v = \left\{ \begin{array}{lll} C \cdot e^{-D_0(\theta)\tau} \cdot \cos(\varsigma\tau - \tau_0), & \varsigma^2 = 1 - D_0^2\left(\theta\right), & D_0\left(\theta\right) < 1; \\ C \cdot ch\left(\varepsilon\tau + \tau_0\right), & \varepsilon^2 = D_0^2\left(\theta\right) - 1, & D_0\left(\theta\right) > 1; \\ e^{-\tau}\left[v_0\left(1 + \tau\right) + v_1\tau\right], & & D_0\left(\theta\right) \to 1, \end{array} \right\}$$

$$(3.5)$$

where C, τ_0, v_0 and v_1 are integration constants.

It is clear from the solutions (3.5) that a generic local anisotropy of space–time (possibly induced from higher dimensions) could play a crucial role in Cosmology. For some prescribed values of nonlinear connection components we can obtain exponential anisotropic acceleration, or damping for corresponding conditions, of the inflational scenarios of universes, for another ones there are possible oscillations.

4 Anisotropic Black Holes and Strings

4.1 Three dimensional la–solutions

We first consider the simplest possible case when (2+1)–dimensional space–time admits a prescribed N–connection structure. The anzats for la–metric (2.5) is chosen in the form

$$\delta s^2 = -N_*^2(r)dt^2 + S_*^2(r)dr^2 + P_*^2(r)\delta y^2, \tag{4.1}$$

where $\delta y = d\varphi + n(r)dr$. The metric (4.1) is written for a la–space with local coordinates $x^0 = t, x^1 = r$ and fiber coordinate $y^1 = \varphi$ and has components: $g_{00} = -N_*^2(r), g_{11} = S_*^2(r)$ and $h_{11} = P_*^2(r)$. The prescription for N-connection from (2.2) is taken $N_0^1 = 0$ and $N_1^1 = n(r)$.

The Einstein equations (2.6) are satisfied if one holds the condition $n = \frac{\ddot{N}_*}{N_*} - \frac{\dot{S}_*}{S_*}$, where, for instance, $\dot{S}_* = \frac{dS_*}{dt}$. So on a (2+1)–dimensional space–time with prescribed generic N–connection there are possible nonsingular la–metrics.

Nevertheless (2+1)–like black hole solutions with singular anisotropies can be constructed, for instance, by choosing the parametrizations

$$P_*^2(r) = P^2(r) = \rho^2(r), \quad N_*^2(r) = N^2(r) = \left(\frac{r}{\rho}\right) \cdot \left(\frac{r^2 - r_+^2}{l}\right), \qquad (4.2)$$

$$S_*^2 = S^2 = \left(\frac{r}{\rho N}\right)^2, \quad n(r) = N^\varphi(r) = -\frac{J}{2\rho^2}$$

where

$$\rho^2 = r^2 + \frac{1}{2}\left(Ml^2 - r_+^2\right), \quad r_+^2 = Ml^2\sqrt{1 - \left(\frac{J}{Ml}\right)^2}$$

and J, M, l are constants characterizing some values of rotational momentum, mass and fundamental length type. In this case the la–metric (4.1) transforms in the well known BTZ–solution for three dimensional black holes [1].

We can also parametrize solutions for la–gravity of type (4.1) as to be equivalent to a locally isotropic anti–de Sitter space with cosmological constant $\Lambda = -\frac{1}{l^2}$ when coefficients (4.2) are modified by the relations $N(r) = N^\perp = f = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{1/2}, N^\varphi(r) = -\frac{J}{2r^2}$, where $M > 0$ and $|J| \leq Ml$ and the solution has an outer event horizon at $r = r_+$ and inner horizon at $r = r_-$, $r_\pm^2 = \frac{Ml^2}{2}\{1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\}$. We conclude that the N–connection could model both singular and nonsingular anisotropies of (2+1)–dimensional space–times.

4.2 Three dimensional la–solutions and strings

We proceed to study the possibility of imbedding of 3–dimensional solutions of la–gravity into the low energy dynamics of la–strings [7, 8, 9].

A la–metric

$$\delta s^2 = -K^{-1}(r)f(r)dt^2 + f^{-1}(r)dr^2 + K(r)\delta y^2, \qquad (4.3)$$

where $\delta y = dx_1 + n(r)dt$, i.e. $N_0^1 = n(r)$ and $N_0^1 = 0$, solves the Einstein la–equations (2.6) if

$$n(r) = \frac{3}{4}\varsigma(r) + \frac{\dot{\varsigma}(r)}{\varsigma} \qquad (4.4)$$

with $\varsigma(r) = \dot{f}/f - \dot{K}/K$, where, for instance $\dot{f} = df/dt$.

Metrics of type (4.3) are considered [5] in an isotropic manner in connection to solutions of type IIA supergravity that describes a non–extremal

intersection of a solitonic 5–brane, a fundamental string and a wave along one of common directions.

We have an anisotropic plane wave solution in $D + 1$ dimensions if

$$\delta s^2 = -K^{-1}(r)f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2 + K(r)\delta y_{(\alpha)}^2 \qquad (4.5)$$

where $\delta y_{(\alpha)} = dx_1 + [1/K'(r) - 1 + \tan\alpha]\,dt$, $K(r) = 1 + \mu^{D-3}\sinh^2\alpha/r^{D-3}$, $(K'(r))^{-1} = 1 - \mu^{D-3}\sinh\alpha\cosh\alpha/(r^{D-3}K)$, $f(r) = 1 - \mu^{D-3}/r^{D-3}$ for isotropic solutions but $f(r)$ is a function defined by the prescribed component of N–connection (4.4) for la–spaces, $r^2 = x_2^2 + ...x_D^2$, and the parameter α define shift translations.

A la–string [7, 8, 9] solution is constructed by including (4.5) into a 10–dimensional la–metric with trivial shift $\delta y_{(\alpha=0)}$

$$\delta s_{(10)}^2 = H_f^{-1}\left[-\frac{f(r)}{K(r)}dt^2 + K(r)\delta y_{(0)}^2\right] +$$

$$dx_2^2 + ... + dx_5^2 + H_{S5}\left[f^{-1}(r)dr^2 + r^2 d\Omega_3^2\right]$$

where the la–string dilaton fields and antisymmetric tensor are defined [5] by the relations $e^{-2\phi} = H_{S5}^{-1}H_f, B_{01} = H_f^{-1} + \tanh\alpha_f, r^2 = x_6^2 + ...x_8^2$, $H_{ijk} = \frac{1}{2}\epsilon_{ijkl}\delta_l H_{S5}$ (in general, one considers la–derivations of type (2.2)) and $i, j, d, l = 6, ..., 9$.

Considering dimensional reductions in variables x_1, x_2, x_3, x_4, x_5 one can construct non–extremal, under singular anisotropies, 5–dimensional la–black hole solutions

$$\delta s_{(5)}^2 = -\lambda^{-2/3}f(r)dt^2 + \lambda^{1/3}\left[f^{-1}(r)dr^2 + r^2 d\Omega_3^2\right]$$

where $\lambda = H_{S5}H_f K = \left(1 + \frac{Q_{S5}}{r^2}\right)\left(1 + \frac{Q_f}{r^2}\right)\left(1 + \frac{Q_K}{r^2}\right); Q_{S5}, Q_f$ and Q_K are constants. Finally we note that for la–backgrounds the function $f(r)$ is connected with the components of N–connection via relation (4.4), i.e. the N–connection structure could model both type of singular (like black hole) and nonsingular locally anisotropic string solutions.

5   Conclusions

The scenario of modelling of physical theories with generic locally anisotropic interactions on nonholonomic bundles provided with nonlinear connection structure has taught us a number of interesting things about a new class of anisotropic cosmological models, black hole solutions and low energy limits of string theories. Generic anisotropy of space–time could be a consequence of reduction from higher to lower dimensions and of quantum filed and space–time structure fluctuations in pre–inflationary period. This way an unification of logical aspects, geometrical background and physical ideas from the generalized Finsler and Kaluza–Klein theories was achieved.

The focus of this paper was to present some exact solutions with prescribed nonlinear connection for the locally anisotropic gravity and string theory. We have shown that a generic anisotropy of Friedmann–Robertson–Walker metrics could result in drastic modifications of cosmological models. It was our task here to point the conditions when the nonlinear connection will model singular, or nonsingular, anisotropies with three dimensional black hole solutions and to investigate the possibility of generalization of such type constructions to string theories.

Acknowledgments

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